

THE ASYMPTOTIC TIAN-YAU-ZELDITCH EXPANSION ON RIEMANN SURFACES WITH CONSTANT CURVATURE

CHIUNG-JU LIU

ABSTRACT. Let M be a regular Riemann surface with a metric which has constant scalar curvature ρ . We give the asymptotic expansion of the sum of the square norm of the sections of the pluricanonical bundles K_M^m . That is,

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|_{h_m}^2 \sim m(1 + \frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \dots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for sufficiently large m .

1. INTRODUCTION

Let M be an n -dimensional compact complex Kähler manifold with an ample line bundle L over M . Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = \text{Ric}(h)$ for some positive Hermitian h metric on L . Such a Kähler metric g is called a polarized Kähler metric. The metric h induces a Hermitian metric h_m on L^m for all positive integers m . Let $\{S_0, \dots, S_{d_m-1}\}$ be an orthonormal basis of the space $H^0(M, L^m)$ with respect to the inner product

$$(1.1) \quad (S, T) = \int_M \langle S(x), T(x) \rangle_{h_m} dV_g,$$

where $d_m = \dim H^0(M, L^m)$ and $dV_g = \frac{\omega_g^n}{n!}$ is the volume form of g . The quantity

$$(1.2) \quad \sum_{i=0}^{d_m-1} \|S_i(x)\|_{h_m}^2$$

is related to the existence of Kähler-Einstein metrics and stability of complex manifolds. A lot of work has been done for (1.2) on compact complex Kähler manifolds. Tian [6] applied Hörmander's L^2 -estimate to produce peak sections and proved the C^2 convergence of the Bergman metrics. Later, Ruan [5] proved the C^∞ convergence. About the same time, Zelditch [7] and Catlin [4] separately generalized the theorem of Tian by showing there is an asymptotic expansion

$$(1.3) \quad \sum_{i=0}^{d_m-1} \|S_i(x)\|_{h_m}^2 \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \dots$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. In [10], Lu proved that each coefficient $a_j(x)$ is a polynomial of the curvature and its covariant derivatives. In particular, $a_1 = \frac{\rho}{2}$, where ρ is the scalar curvature of M . These polynomials can be

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found by finitely many steps of algebraic operations. Recently, Song [3] generalized Zelditch's theorem on orbifolds of finite isolated singularities. The information on the singularities can be found in the expansion.

On the Riemann surfaces with bounded curvature, Lu [9] proved that there is a lower bound for (1.2). Later, the result of Lu and Tian [8] implies that on the Riemann surfaces with constant scalar curvature ρ , the asymptotic expansion (1.3) is given by

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|_{h_m}^2 \sim m(1 + \frac{\rho}{2m}) + O\left(\frac{1}{m^p}\right)$$

for any $p > 0$. In the current paper, we obtain a more precise result for (1.3).

Theorem 1.1. *Let M be a regular compact Riemann surface and K_M be the canonical line bundle endowed with a Hermitian metric h such that the curvature $\text{Ric}(h)$ of h defines a Kähler metric g on M . Suppose that this metric g has constant scalar curvature ρ . Then there is a complete asymptotic expansion:*

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|_{h_m}^2 \sim m(1 + \frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \dots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for some $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$, where δ is the injective radius at x_0 .

Our result indicates that the asymptotic expansion (1.3) is in exponential decay. Engliš [2] has an asymptotically expansion for the Berezin transformation on any planar domain of hyperbolic type. He also showed that Berezin kernel [1] has

$$\tilde{B}(\eta, \eta) = m \left(1 + O(1)\rho_0(0)^{\frac{\pi m-3}{2}}\right),$$

where $\rho_0(0)$ is a positive constant.

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2. GENERAL SET UP

Let M be an n -dimensional compact complex Kähler manifold with a polarized line bundle $(L, h) \rightarrow M$. Choose the K -coordinates (z_1, \dots, z_n) on an open neighborhood U of a fixed point $x_0 \in M$. Then the Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta=1}^n g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

satisfies

$$(2.1) \quad g_{\alpha\bar{\beta}}(x_0) = \delta_{\alpha\bar{\beta}}, \quad \frac{\partial^{p_1+\dots+p_n} g_{\alpha\bar{\beta}}}{\partial z_1^{p_1} \dots \partial z_n^{p_n}}(x_0) = 0,$$

for $\alpha, \beta = 1, \dots, n$ and any nonnegative integers p_1, \dots, p_n with $p_1 + \dots + p_n \neq 0$.

We also choose a local holomorphic frame e_L of the line bundle L at x_0 such that a is the local representation function of the Hermitian metric h . That is,

$$\text{Ric}(h) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log a.$$

Under the K -coordinate, the function a has the properties

$$(2.2) \quad a(x_0) = 1, \quad \frac{\partial^{p_1+\dots+p_n}}{\partial z_1^{p_1} \dots \partial z_n^{p_n}}(a)(x_0) = 0$$

for any nonnegative integers p_1, \dots, p_n with $p_1 + \dots + p_n \neq 0$.

Let $\{S_0, \dots, S_{d_m-1}\}$ be a basis of $H^0(M, L^m)$. Assume that at the point $x_0 \in M$,

$$S_0(x_0) \neq 0, \quad S_i(x_0) = 0, \quad i = 1, \dots, d_m - 1.$$

If the set $\{S_0, \dots, S_{d_m-1}\}$ is not an orthonormal basis, we may do the following:
Let the metric matrix

$$F_{ij} = (S_i, S_j), \quad i, j = 0, \dots, d_m - 1$$

with respect to the inner product (1.1). By definition, (F_{ij}) is a positive definite Hermitian matrix. We can find a $d_m \times d_m$ matrix G_{ij} such that

$$F_{ij} = \sum_{k=0}^{d_m-1} G_{ik} \overline{G_{jk}}.$$

Let (H_{ij}) be the inverse of (G_{ij}) . Then $\{\sum_{j=0}^{d_m-1} H_{ij} S_j\}$ forms an orthonormal basis of $H^0(M, L^m)$. The left hand side of (1.2) is equal to

$$(2.3) \quad \sum_{i=0}^{d_m-1} \left\| \sum_{j=0}^{d_m-1} H_{ij} S_j(x_0) \right\|_{h_m}^2 = \sum_{i=0}^{d_m-1} |H_{i0}|^2 \|S_0(x_0)\|_{h_m}^2.$$

Let (I_{ij}) be the inverse matrix of (F_{ij}) . Denote that

$$(2.4) \quad \sum_{i=0}^{d_m-1} |H_{i0}|^2 = I_{00}.$$

In order to compute (2.4), we need a suitable choice of the basis $\{S_0, \dots, S_{d_m-1}\}$. We select some of Tian's peak sections in our basis. The following lemma an improved version of Tian's result [6, Lemma 1.2], which is done by Lu and Tian.

Let \mathbb{Z}_+^n be the set of n -tuple integers $P = (p_1, \dots, p_n)$ such that each p_i is a nonnegative integer for $i = 1, \dots, n$. For $P \in \mathbb{Z}_+^n$, we denote that $z^P = z_1^{p_1} \dots z_n^{p_n}$ and $|P| = p_1 + \dots + p_n$.

Lemma 2.1 (Tian). *Suppose $\text{Ric}(g) \geq -K\omega_g$, where $K > 0$ is a constant. For $P \in \mathbb{Z}_+^n$ and an integer $p' > |P|$, let m be an integer such that*

$$m > \max\{e^{20\sqrt{n+2p'}} + 2K, e^{8(p'-1+n)}\}.$$

Then there is a holomorphic section $S_{P,m} \in H^0(M, L^m)$, satisfying

$$(2.5) \quad \left| \int_M \|S_{P,m}\|_{h_m}^2 dV_g - 1 \right| \leq C e^{-\frac{1}{8}(\log m)^2}.$$

Moreover, $S_{P,m}$ can be decomposed as

$$S_{P,m} = \tilde{S}_{P,m} - u_{P,m}$$

such that

$$(2.6) \quad \tilde{S}_{P,m}(x) = \lambda_P \eta \left(\frac{m|z|^2}{(\log m)^2} \right) z^P e_L^m = \begin{cases} \lambda_P z^P e_L^m & x \in \{|z| \leq \frac{\log m}{\sqrt{2m}}\}, \\ 0 & x \in M \setminus \{|z| \leq \frac{\log m}{\sqrt{m}}\}, \end{cases}$$

and

$$(2.7) \quad \int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq C e^{-\frac{1}{4}(\log m)^2},$$

where η is a smoothly cut-off function

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{for } t \geq 1. \end{cases}$$

satisfying $0 \leq -\eta'(t) \leq 4$ and $|\eta''(t)| \leq 8$ and

$$(2.8) \quad \lambda_P^{-2} = \int_{|z| \leq \frac{\log m}{\sqrt{m}}} |z^P|^2 a^m dV_g.$$

Proof. Define the weight function

$$\Psi(z) = (n + 2p')\eta\left(\frac{m|z|^2}{(\log m)^2}\right) \log\left(\frac{m|z|^2}{(\log m)^2}\right).$$

A straightforward computation gives

$$(2.9) \quad \sqrt{-1}\partial\bar{\partial}\Psi \geq -\frac{100m(n+2p')}{(\log m)^2}\omega_g.$$

By using (2.9), we can verify that

$$\langle \partial\bar{\partial}\Psi + \frac{2\pi}{\sqrt{-1}}(\text{Ric}(h^m) + \text{Ric}(g)), v \wedge \bar{v} \rangle_g \geq \frac{1}{4}m\|v\|_g^2.$$

For $P \in \mathbb{Z}_+^n$, consider the 1-form

$$w_P = \bar{\partial}\eta\left(\frac{m|z|^2}{(\log m)^2}\right)z^P e_L^m.$$

Since $w_P \equiv 0$ in a neighborhood of x_0 , we have

$$\int_M \|w_P\|_{h_m}^2 e^{-\Psi} dV_g < +\infty.$$

By [6, Prop. 2.1], there exists a smooth L^m -valued section u_P such that $\bar{\partial}u_P = w_P$ and

$$(2.10) \quad \int_M \|u_P\|_{h_m}^2 e^{-\Psi} dV_g \leq \frac{4}{m} \int_M \|w_P\|_{h_m}^2 e^{-\Psi} dV_g < \infty.$$

By direct computation, we get

$$\int_M \|u_P\|_{h_m}^2 e^{-\Psi} dV_g \leq \frac{C(\log m)^{2(p-1)}}{m^p} \int_{\frac{\log m}{\sqrt{2m}} \leq |z| \leq \frac{\log m}{\sqrt{m}}} a^m dV_0.$$

Under the K -coordinate, we have

$$a^m = e^{m \log a} = e^{m(-|z|^2 + O(|z|^4))}.$$

Hence we get

$$\int_M \|u_P\|_{h_m}^2 e^{-\Psi} dV_g \leq \frac{C_1(\log m)^{2(p-1+n)}}{m^{p+n}} e^{-\frac{1}{2}(\log m)^2}$$

for some constant C_1 . Let $\tilde{S}_{P,m} = \lambda_P \eta\left(\frac{m|z|^2}{(\log m)^2}\right)z^P e_L^m$ and $u_{P,m} = \lambda_P u_P$. Use a result in [10]

$$\lambda_P^2 \leq C_2 m^{n+|P|}$$

for some constant C_2 . Then we have

$$\int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq C(\log m)^{2(|P|-1+n)} e^{-\frac{1}{2}(\log m)^2}.$$

Choosing $m > e^{8(p'-1+n)}$, we obtain

$$\int_M \|u_{P,m}\|_{h_m}^2 dV_g \leq C e^{-\frac{1}{4}(\log m)^2}.$$

□

3. PROOF OF THEOREM 1.1

Proof. Let M be a smooth compact Riemann surface with a metric g that has constant scalar curvature. Let x_0 be a fixed point. Let

$$U = \{x : \text{dist}(x, x_0) < \delta\},$$

where δ is the injective radius at x_0 . It is well known that on a Riemann surface there is an isothermal coordinate at each point on U . We may assume that there is a holomorphic function z on U and it defines the conformal structure on U . That is,

$$ds^2 = g dz d\bar{z}$$

and $g > 0$. The metric g satisfies

$$(3.1) \quad \Delta \log g = -\rho, \quad g(x_0) = 1, \quad \text{and} \quad \frac{\partial g}{\partial z}(x_0) = 0,$$

where

$$\Delta = g^{-1} \frac{\partial^2}{\partial z \partial \bar{z}}$$

is the complex Laplace of M . Since the metric g has conformal structure, it is rotationally symmetric. We can write (3.1) in polar coordinates (r, θ) :

$$(3.2) \quad \frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} - \frac{1}{g} \left(\frac{\partial g}{\partial r} \right)^2 = -4\rho g^2, \quad g(0, \theta) = 1, \quad \frac{\partial g}{\partial r}(0, \theta) = 0,$$

where $z = r e^{i\theta}$, and $|z|^2 = r^2$. There exists a solution

$$(3.3) \quad g = \frac{1}{(1 + \frac{\rho}{2}|z|^2)^2}$$

to (3.2) for $|z| < \sqrt{-\frac{2}{\rho}}$ if $\rho < 0$. Suppose that there exists another solution g_1 to (3.2). We have

$$\Delta \log(g_1/g) = 0 \quad \text{and} \quad g_1(x_0) = 1.$$

For $\rho < 0$, let $r_0 < \sqrt{-\frac{2}{\rho}}$. Since g and g_1 are rotationally symmetric, they remain constant on $|z| = r_0$. The harmonic function $\log(g/g_1)$ is a constant on $|z| \leq r_0$ by Maximum Principle. By definition, we have $g(x_0) = g_1(x_0) = 1$. Therefore, the solution in (3.3) is unique around x_0 . By the same reason, $g = g_1$ on $\{\text{dist}(x, x_0) \leq \delta_1\}$ for some $\delta_1 < \delta$ for $\rho \leq 0$.

Let a be the local representation of the metric h on K_M such that

$$-\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log a = \omega_g.$$

If we normalize a and a satisfies

$$(3.4) \quad \triangle \log a = -1, \quad a(x_0) = 1, \quad \frac{\partial a}{\partial z}(x_0) = 0.$$

Since

$$-\frac{\partial^2}{\partial z \partial \bar{z}} \log a = g,$$

$\log a$ is also rotationally symmetric. Since

$$(3.5) \quad a = \begin{cases} (1 + \frac{\rho}{2}|z|^2)^{-\frac{2}{\rho}}, & \text{if } \rho \neq 0; \\ e^{-|z|^2}, & \text{if } \rho = 0. \end{cases}$$

satisfies (3.4), the local uniqueness is due to the same reason.

We need to choose sufficient large m such that $\frac{\log m}{\sqrt{m}} < \min\{\delta, \sqrt{\frac{2}{|\rho|}}\}$. With these particular solutions of g and a , we further compute

$$(3.6) \quad \begin{aligned} \lambda_0^{-2} &= \int_{|z| \leq \frac{\log m}{\sqrt{m}}} a^m g \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} \\ &= 2 \int_0^{\frac{\log m}{\sqrt{m}}} (1 + \frac{\rho}{2}r^2)^{-\frac{2m}{\rho}-2} r dr \\ &= \frac{1}{m + \frac{\rho}{2}} \left(1 - (1 + \frac{\rho(\log m)^2}{2m})^{-1-\frac{2m}{\rho}} \right) \quad \text{for } \rho \neq 0. \end{aligned}$$

For $m > \max\{|\rho|^{4/3}, 10\}$, we have $|\frac{\rho(\log m)^2}{2m}| < 1/2$. For $\rho \neq 0$, this gives

$$\left(1 + \frac{\rho(\log m)^2}{2m}\right)^{-1-\frac{2m}{\rho}} \leq 2e^{-\frac{2m}{\rho}\left(\frac{\rho}{2}\frac{(\log m)^2}{m} - \frac{1}{2}\left(\frac{\rho}{2}\frac{(\log m)^2}{m}\right)^2 + \dots\right)} \leq Ce^{-(\log m)^2}.$$

For $\rho = 0$, we have

$$\lambda_0^{-2} = \int_{|z| \leq \frac{\log m}{\sqrt{m}}} e^{-m|z|^2} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = \frac{1}{m} (1 + O(e^{-(\log m)^2})).$$

Hence we obtain

$$(3.7) \quad \lambda_0^{-2} = \frac{1}{m + \frac{\rho}{2}} \left(1 + O(e^{-(\log m)^2}) \right).$$

From the properties of g and a , the isothermal coordinate (U, z) is a K -coordinate. According to Lemma 2.1, we may choose two peak sections

$$\begin{aligned} S_{0,m} &= \lambda_0 \left(\eta \left(\frac{m|z|^2}{(\log m)^2} \right) (dz)^m - u_0 \right) \\ S_{1,m} &= \lambda_1 \left(\eta \left(\frac{m|z|^2}{(\log m)^2} \right) z (dz)^m - u_1 \right) \end{aligned}$$

in $H^0(M, K_M^m)$ for some $m > e^{20\sqrt{1+4}} + 2|\rho|$. Obviously, we have $S_{0,m}(x_0) \neq 0$ and $S_{1,m}(x_0) = 0$. Let the subspace

$$V = \{S \in H^0(M, K_M^m) | S(x_0) = 0, DS(x_0) = 0\},$$

where D is a covariant derivative on K_M^m . Let T_1, \dots, T_{d_m-2} be an orthonormal basis of V . Let

$$(3.8) \quad S_i = \begin{cases} S_{i,m} & \text{if } i = 0, 1 \\ T_{i-1} & \text{if } 2 \leq i \leq d_m - 1 \end{cases}.$$

Then $\{S_i\}_{i=0}^{d_m-1}$ forms a basis for $H^0(M, K_M^m)$. Locally, each T_i has the form $f_i(dz)^m$ for some holomorphic function f_i defined in U . The holomorphic function f_i has Taylor expansion as $f_i = \sum_{\alpha=2}^{\infty} b_{i\alpha} z^\alpha$ in U , since $T_i(x_0) = 0$ and $DT_i(x_0) = 0$ for $1 \leq i \leq d_m - 2$

Lemma 3.1. *Let $\{S_i\}_{i=0}^{d_m-1}$ be the basis of $H^0(M, K_M^m)$, defined in (3.8). For $m > e^{20\sqrt{5}} + 2|\rho|$, the Hermitian matrix*

$$(S_i, S_j) = \int_M \langle S_i(x), S_j(x) \rangle_{h_m} dV_g$$

is given by

$$\begin{aligned} (S_0, S_0) &= 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_0, S_1) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_1, S_1) &= 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_0, S_i) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_1, S_i) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\ (S_i, S_j) &= \delta_{ij} \end{aligned}$$

for $i, j = 2, \dots, d_m - 1$.

Proof. By definition, we have $(S_i, S_j) = \delta_{ij}$ for $2 \leq i, j \leq d_m - 1$. The inner product of (S_i, S_i) for $0 \leq i \leq 1$ is directly from Lemma 2.1. Since $a^m g$ is rotationally symmetric, we have

$$\int_{|z| \leq \frac{\log m}{\sqrt{m}}} \bar{z}^\alpha a^m g dV_0 = 0 \quad \text{for arbitrary positive integer } \alpha.$$

Then we get

$$\begin{aligned} (S_0, S_1) &= (\tilde{S}_0, \tilde{S}_1) + (\lambda_0 u_0, \tilde{S}_1) + (\tilde{S}_0, \lambda_1 u_1) + (u_0, u_1) \\ &= O\left(e^{-\frac{(\log m)^2}{8}}\right). \end{aligned}$$

Consider

$$\begin{aligned} (S_0, S_i) &= \int_M \langle \lambda_0 \left(\eta \left(\frac{m|z|^2}{(\log m)^2} \right) (dz)^m - u_0 \right), f_{i-1}(dz)^m \rangle_{h_m} dV_g \\ &\leq \lambda_0 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \sum_{\alpha=2}^{\infty} b_{(i-1)\alpha} \bar{z}^\alpha a^m g dV_0 + \lambda_0 \|u_0\| \cdot \|S_i\|. \end{aligned}$$

Thus we have

$$(S_0, S_i) = O\left(e^{-\frac{(\log m)^2}{8}}\right) \quad \text{for } 2 \leq i \leq d_m - 1.$$

Similarly, consider

$$(S_1, S_j) \leq \lambda_0 \int_{|z| \leq \frac{\log m}{\sqrt{m}}} \sum_{\alpha=2}^{\infty} b_{(i-1)\alpha} z^\alpha \bar{z}^\alpha a^m g dV_0 + \lambda_1 \|u_1\| \cdot \|S_i\| \quad \text{for } 2 \leq i \leq d_m - 1.$$

Since $a^m g$ is rotationally symmetric, $\int_{|z| \leq \frac{\log m}{\sqrt{m}}} z \bar{z}^\alpha a^m g dV_0 = 0$ for $\alpha \geq 2$. Hence we obtain

$$(S_1, S_i) = O\left(e^{-\frac{(\log m)^2}{8}}\right).$$

□

According to [10, Definition 3.1], the metric matrix (F_{ij}) can be represented by the block matrices

$$(3.9) \quad (F_{ij}) = \begin{pmatrix} (S_0, S_0) & (S_0, S_1) & M_{13} \\ (S_1, S_0) & (S_1, S_1) & M_{23} \\ M_{31} & M_{32} & E \end{pmatrix},$$

where $M_{13} = ((S_0, S_2), \dots, (S_0, S_{d_m-1}))$, $M_{23} = ((S_1, S_2), \dots, (S_1, S_{d_m-1}))$, $M_{31} = \overline{M_{13}^T}$, $M_{32} = \overline{M_{23}^T}$, and E is a $(d_m - 2) \times (d_m - 2)$ identity matrix. By using [10, Lemma 3.1], we obtain

$$(3.10) \quad I_{00} = \frac{1}{(S_0, S_0)} + \left(\frac{1}{(S_0, S_0)}\right)^2 \begin{pmatrix} (S_0, S_1) & M_{13} \end{pmatrix} \tilde{M}^{-1} \begin{pmatrix} (S_1, S_0) \\ M_{31} \end{pmatrix},$$

where

$$\tilde{M} = \begin{pmatrix} (S_1, S_1) & M_{23} \\ M_{32} & E \end{pmatrix} - \frac{1}{(S_0, S_0)} \begin{pmatrix} (S_1, S_0) \\ M_{31} \end{pmatrix} \begin{pmatrix} (S_0, S_1) & M_{13} \end{pmatrix}.$$

Applying Lemma 3.1 in (3.10), we get

$$(3.11) \quad I_{00} = 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right).$$

In order to evaluate the expansion of (2.3), we are left to find $\|S_0(x_0)\|_{h_m}^2 = \lambda_0^2$. From (3.7), we have

$$\lambda_0^2 = m\left(1 + \frac{\rho}{2m}\right) \left(1 + O(e^{-(\log m)^2})\right).$$

Therefore, the Tian-Yau-Zelditch expansion according to (2.3) on a Riemann surface with constant scalar curvature ρ is

$$\begin{aligned} I_{00}\lambda_0^2 &= (1 + O\left(e^{-\frac{(\log m)^2}{8}}\right))m\left(1 + \frac{\rho}{2m}\right) \left(1 + O(e^{-(\log m)^2})\right) \\ &= m\left(1 + \frac{\rho}{2m}\right) + O\left(e^{-\frac{(\log m)^2}{8}}\right) \end{aligned}$$

for $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$. □

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E-mail address: `cjliu4@ntu.edu.tw`

TAIDA INSTITUTE OF MATHEMATICAL SCIENCE, NATIONAL TAIWAN UNIVERSITY, TAIWAN 106